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An application to the MKZ-operators of generalized convexity on ECT-systems

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Abstract

We analyze some relations between classical convexities and some generalized convexities. The results are applied to obtain some new shape preserving properties of the Meyer-König and Zeller operators.

Keywords: Convexities of order k; Absolutely monotonic functions; Completely monotonic functions; Meyer-König and Zeller operators; Baskakov operators

1. Introduction

We begin establishing some notations. We denote by $D^k f$ the kth derivative of a function f. Let $\Delta_h^k f(x) = \sum_{j=0}^{k} \binom{k}{j}(-1)^{k-j} f(x+jh)$ be the classical finite difference and $[t_0, \ldots, t_k; f]$ be the usual divided difference of the function f. We will also use the notation $f^{(0)} = \Delta_h^0 f = f$. Let I be an interval. A function $f \in \mathbb{R}^I$ is k-convex ($k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$) if $[t_0, \ldots, t_k; f] \geqslant 0$ at arbitrary $k+1$ distinct points in I. If $f \in C^k(I)$, f is convex of order k if, and only if, $D^k f \geqslant 0$ on I. A real valued function f defined on I is said to be absolutely monotonic (of order k) if it is convex of order s, for all $0 \leqslant s$ $(\leqslant k)$. Similarly, a function $f \in C^k(I)$ is said to be completely monotonic (of order k) if $(-1)^s D^s f \geqslant 0$ on I, for all $0 \leqslant s$ $(\leqslant k)$.

The paper is organized as follows. In Section 2 we present some usual definitions of relative differentiation [5,9] and generalized convexity [4,5] and we prove that they are equivalent. Section 3 contains the main result. It is proved that M_n preserves φ-convex functions of all order (see Definition 4), where M_n is the Meyer-König and Zeller operator of order n and $\varphi(t) = t/(1-t)$. Moreover, we obtain a very large class of functions f (which contains the power functions φ^k, with $k \in \mathbb{N}$) such that $M_n f$ is an absolutely monotonic function, for all $n \in \mathbb{N}$.

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2. Relative differentiation and generalized convexity

Definition 1. (See [9].) Let $P_n := \{p_0, \ldots, p_n\}$ be a sequence of functions defined on an open interval I, with $p_0 > 0$ and $p_1, \ldots, p_n$ strictly increasing and continuous. Let f be a real-valued function defined on I, and, for $x \in I$, let $D^0_{P_n} f(x) := f(x)/p_0(x)$ and, provided the limits exist,

$$D^k_{P_n} f(x) := \lim_{h\to 0} \frac{D^{k-1} f(x+h) - D^{k-1} f(x)}{p_k(x+h) - p_k(x)}, \quad 1 \leqslant k \leqslant n.$$

If $D^k_{P_n} f(x)$ exists for every $k = 0, \ldots, n$, then f is called relatively differentiable with respect to P_n at x. We will denote by $D(P_n, I)$ the set of all relatively differentiable functions with respect to P_n at every $x \in I$.

It is obvious that f is n times differentiable (in the usual sense) if and only if it is relatively differentiable with respect to $P_n = \{1, e_1, \ldots, e_1\}$, where e_1 denotes the identity map and, in such a case, $D^k_{P_n} f = D^k f$. When $P_n = P^\varphi_n := \{1, \varphi, \ldots, \varphi^n\}$ where $\varphi(t) = t/(1-t)$, the notion of relative differentiation with respect to P^φ_n is related to the notion of φ-derivative introduced in [5].

Definition 2. (See [5].) For $k \in \mathbb{N}_0$, $h > 0$, $x \in [0, 1)$ and $f \in \mathbb{R}^{[0,1)}$, we define (whenever it make sense) $D^k_\varphi f(x) = D^k(f \circ \varphi^{-1})(\varphi(x))$ and $\Delta^k_{\varphi,h} f(x) = \Delta^k_h(f \circ \varphi^{-1})(\varphi(x))$.

Theorem 3. *A function $f \in D(P^\varphi_n, (0, 1))$ if, and only if, f is n times differentiable in $(0, 1)$ and, in this case,*

$$D^k_{P^\varphi_n} f(x) = D^k_\varphi f(x) = \lim_{h\to 0} \frac{\Delta^k_{\varphi,h} f(x)}{h^k}, \quad x \in (0, 1),\ k = 0, \ldots, n.$$

Proof. Of course, $D^0_{P^\varphi_n} f = D^0_\varphi = f$. For $x \in (0, 1)$,

$$D_{P^\varphi_n} f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{\varphi(x+h) - \varphi(x)} = (1-x)^2 Df(x) = D_\varphi f(x).$$

By induction on k,

$$\begin{aligned} D^k_\varphi f(x) &= D_\varphi D^{k-1}_\varphi f(x) = D_\varphi D^{k-1}_{P^\varphi_n} f(x) = D\big(D^{k-1}_{P^\varphi_n} f \circ \varphi^{-1}\big)\big(\varphi(x)\big) \\ &= \lim_{h\to 0} \frac{(D^{k-1}_{P^\varphi_n} f \circ \varphi^{-1})(\varphi(x) + h) - (D^{k-1}_{P^\varphi_n} f \circ \varphi^{-1})(\varphi(x))}{h} \\ &= \lim_{h\to 0} \frac{D^{k-1}_{P^\varphi_n} f(x + \frac{h(1-x)^2}{1+h(1-x)}) - D^{k-1}_{P^\varphi_n} f(x)}{h}. \end{aligned}$$

Changing $h = \frac{h'}{(1-x)(1-x-h')} = \varphi(x+h') - \varphi(x)$, we get $D^k_\varphi f(x) = D^k_{P^\varphi_n} f(x)$. $\square$

Popoviciu [8] introduced, for any Extended Complete Tchebycheff system (ECT-system) $\mathcal{U} = \{u_0, \ldots, u_k\}$ on I, the generalized divided differences as a ratio of two determinants

$$\begin{bmatrix} u_0 & \cdots & u_k \\ x_0 & \cdots & x_k \end{bmatrix}; f \Bigg] = \frac{\det(u_0, \ldots, u_{k-1}, f; x_0, \ldots, x_k)}{\det(u_0, \ldots, u_k; x_0, \ldots, x_k)}, \tag{1}$$

where $\det(g_0, \ldots, g_k; x_0, \ldots, x_k) = \det(g_i(x_j))_{i,j=0,\ldots,k}$, for $x_0 < \cdots < x_k$ in I. If (1) is nonnegative for all $x_0 < \cdots < x_k$ in I, the function f is said to be convex on I with respect to $\mathcal{U}$ (see [4]). When $u_i(x) = x^i$, $0 \leqslant i \leqslant k$, we obtain from (1) the classical divided differences $[x_0, \ldots, x_k; f]$ and so the classical definition of k-convexity. On the other hand, it is easy to show that $P^\varphi_k = \{1, \varphi, \ldots, \varphi^k\}$ is an ECT-system on $[0, 1)$ and that

$$\begin{bmatrix} 1 & \varphi & \cdots & \varphi^k \\ x_0 & x_1 & \cdots & x_k \end{bmatrix} ; f \Bigg] = \big[\varphi(x_0), \dots, \varphi(x_k); f \circ \varphi^{-1}\big]. \tag{2}$$

So we can consider the notion of convexity with respect to P_k^{φ}. In the next we related it with another notion of generalized convexity introduced in [5].

Definition 4. Let $k \in \mathbb{N}_0$. A function $f \in \mathbb{R}^{[0,1)}$ is said to be φ-convex of order k if $\Delta^k_{\varphi,h} f \geqslant 0$, for all $h > 0$.

Theorem 5. *Let $\varphi(t) = t/(1-t)$, $f \in \mathbb{R}^{[0,1)}$ and $k \in \mathbb{N}_0$. The following conditions are equivalent*: (a) *f is convex with respect to P_k^{φ}*; (b) *$f \circ \varphi^{-1}$ is convex of order k. If, in addition, f is continuous, then* (a) *and* (b) *are also equivalent to condition* (c) *f is φ-convex of order k.*

Proof. (a) $\Leftrightarrow$ (b) follows from (2) and the fact that φ is positive and strictly increasing on $[0, 1)$. The assertion (b) $\Leftrightarrow$ (c) follows from the identity (see [3, p. 121, (7.9)])

$$\Delta^k_{\varphi,h} f(x) = k!h^k\big[\varphi(x), \varphi(x) + h, \dots, \varphi(x) + kh; f \circ \varphi^{-1}\big].$$

If $f \in C^k[0, 1)$ then $[\varphi(x_0), \dots, \varphi(x_k); f \circ \varphi^{-1}] = D^k(f \circ \varphi^{-1})(t_k) = D^k_{\varphi} f(\xi_k)$ for some $\varphi(x_0) < t_k < \varphi(x_k)$ and $\xi_k = \varphi^{-1}(t_k)$. Hence, taking into account (2) we obtain the following result. $\square$

Corollary 6. *For $k \in \mathbb{N}_0$ and $\varphi(t) = t/(1-t)$, a function $f \in C^k[0, 1)$ is φ-convex of order k if and only if $D^k_{\varphi} f \geqslant 0$.*

Theorem 7. *Let $k \in \mathbb{N}$ and $f \in C^k[0, 1)$.*

(i) *If $f \circ \varphi^{-1}$ is absolutely monotonic* (*of order k*)*, then f is absolutely monotonic* (*of order k*).
(ii) *If f is completely monotonic* (*of order k*)*, then $f \circ \varphi^{-1}$ is completely monotonic* (*of order k*).

Proof. The proof follows from the equalities

$$D^k f(x) = \sum_{i=1}^{k} \binom{k}{i} \frac{(k-1)!}{(i-1)!} (1-x)^{-(k+i)} D^i_{\varphi} f(x)$$

and

$$D^k_{\varphi} f(x) = \sum_{i=1}^{k} \binom{k}{i} \frac{(k-1)!}{(i-1)!} (x-1)^{k+i} D^i f(x),$$

which can be checked by induction. $\square$

3. Application: Simultaneous shape preserving properties of Meyer-König and Zeller operators

Now we are interested in to study the shape preserving properties of the Meyer-König and Zeller operators (see (6)). It is not an easy task. Only partial results are known (see [1,6]). For instance, Lupas [6] shows that M_n operators preserve i-convexity, for $i = 0, 1, 2$. But it is not true for 3-convexity. We are interested in establishing conditions to assure the joint preservation of all classical convexities. It is clear that composition of two positive linear operators gives place to a positive linear operator. This idea can be used to transform operators defined on $[0, \infty)$ into another one defined on $[0, 1)$. Here, we take advantage of it to obtain results concerning the shape preserving properties of some operators, with the help of similar properties of the associated operators.

For $g \in \mathbb{R}^{[0,\infty)}$ and $y \in [0, \infty)$, the Baskakov operators are defined by

$$V_n(g, y) = \frac{1}{(1+y)^n} \sum_{k=0}^{\infty} \binom{k+n-1}{k} \left(\frac{y}{1+y}\right)^k g\left(\frac{k}{n}\right), \tag{3}$$

whenever $|V_n(g, y)| < \infty$. Let us denote $\mathcal{D}[0, \infty) = \{g \in \mathbb{R}^{[0,\infty)}\colon$ for each $n \in \mathbb{N}$ and $y \in [0, \infty)$, $|V_n(g, y)| < \infty\}$.

Define mappings $U:\mathbb{R}^{[0,\infty)}\to\mathbb{R}^{[0,1)}$ and $U^{-1}:\mathbb{R}^{[0,1)}\to\mathbb{R}^{[0,\infty)}$

$$U(g,x)=(1-x)g\bigl(\varphi(x)\bigr),\quad x\in[0,1),\tag{4}$$

$$U^{-1}(f,y)=(1+y)f\bigl(\varphi^{-1}(y)\bigr),\quad y\in[0,\infty),\tag{5}$$

where $\varphi(t)=t/(1-t)$. Set $\mathcal{D}[0,1)=\{f\in\mathbb{R}^{[0,1)}\colon\ f=U(g),\ g\in\mathcal{D}[0,\infty)\}$. For $f\in\mathcal{D}[0,1)$ and $x\in[0,1)$, set

$$M_n=U\circ V_n\circ U^{-1}.$$

If $f\in\mathcal{D}[0,1)$, $x\in[0,1)$ and $y=\varphi(x)$, then

$$\begin{aligned}M_n(f,x)&=(1-x)V_n\bigl(U^{-1}f,y\bigr)=\frac{1-x}{(1+y)^n}\sum_{k=0}^{\infty}\binom{k+n-1}{k}\left(\frac{y}{1+y}\right)^k U^{-1}f\left(\frac{k}{n}\right)\\&=(1-x)^{n+1}\sum_{k=0}^{\infty}\binom{k+n}{k}x^k f\left(\frac{k}{n+k}\right)\end{aligned}\tag{6}$$

and we obtain the Meyer-König and Zeller (in what follows MKZ) operators presented by Cheney and Sharma in [2] as a modification of the operators defined in [7]. Notice that, for $t\in[0,\infty)$,

$$\begin{aligned}\Delta_{1/n}U^{-1}f(t)&=\bigl(\Delta_{1/n}\bigl[(1+e_1)\bigl(f\circ\varphi^{-1}\bigr)\bigr]\bigr)(t)\\&=\Delta_{1/n}\bigl(f\circ\varphi^{-1}\bigr)(t)+\left(t+\frac{1}{n}\right)\bigl(f\circ\varphi^{-1}\bigr)\left(t+\frac{1}{n}\right)-t\bigl(f\circ\varphi^{-1}\bigr)(t)\\&=\left(1+t+\frac{1}{n}\right)\Delta_{1/n}\bigl(f\circ\varphi^{-1}\bigr)(t)+\frac{1}{n}\bigl(f\circ\varphi^{-1}\bigr)(t).\end{aligned}$$

Therefore

$$n(1+t)\Delta_{1/n}U^{-1}f(t)=U^{-1}\bigl[\bigl((n+ne_1+1)\Delta_{1/n}\bigl(f\circ\varphi^{-1}\bigr)\bigr)\circ\varphi\bigr](t)+U^{-1}f(t).$$

By induction, it can be proved that, for $j\in\mathbb{N}$,

$$\Delta^j_{1/n}U^{-1}f(t)=\left(1+t+\frac{j}{n}\right)\Delta^j_{1/n}\bigl(f\circ\varphi^{-1}\bigr)(t)+\frac{j}{n}\Delta^{j-1}_{1/n}\bigl(f\circ\varphi^{-1}\bigr)(t)\tag{7}$$

and

$$\begin{aligned}n\bigl(1+e_1(t)\bigr)\Delta^j_{1/n}U^{-1}f(t)&=U^{-1}\bigl((n+ne_1+j)\bigl[\Delta^j_{1/n}\bigl(f\circ\varphi^{-1}\bigr)\circ\varphi\bigr](t)\bigr)\\&\quad+jU^{-1}\bigl[\bigl(\Delta^{j-1}_{1/n}\bigl(f\circ\varphi^{-1}\bigr)\bigr)\circ\varphi\bigr](t).\end{aligned}\tag{8}$$

It is known that if $i\in\mathbb{N}_0$, then

$$D^iV_n(g,y)=\sum_{k=0}^{\infty}\frac{(n+k+i-1)!}{(n-1)!k!}\Delta^i_{1/n}g\left(\frac{k}{n}\right)\frac{y^k}{(1+y)^{n+k+i}}\tag{9}$$

and

$$V_n(g,y)=\sum_{k=0}^{\infty}\binom{n+k-1}{k}\Delta^k_{1/n}g(0)y^k.$$

From this we obtain a new representation for MKZ operators.

Proposition 8. *If $f\in\mathcal{D}[0,1)$ and $x\in[0,1)$ ($\varphi(x)=x/(1-x)$), then*

$$M_n(f,x)=\sum_{k=0}^{\infty}\binom{n+k}{k}\Delta^k_{1/n}\bigl(f\circ\varphi^{-1}\bigr)(0)\left(\frac{x}{1-x}\right)^k.$$

Proof. From (7), we obtain, for $x \in [0,1)$,

$$\begin{aligned}
M_n(f,x) &= \big(U \circ V_n \circ U^{-1}\big)(f)(x) = (1-x)\big(V_n \circ U^{-1}(f)\big)\big(x/(1-x)\big) \\
&= (1-x)\sum_{k=0}^{\infty}\binom{n+k-1}{k}\big(\Delta_{1/n}^k U^{-1}(f)\big)(0)\left(\frac{x}{1-x}\right)^k \\
&= (1-x)\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{n+k}{n}\Delta_{1/n}^k\big(f\circ\varphi^{-1}\big)(0)\left(\frac{x}{1-x}\right)^k \\
&\quad + (1-x)\sum_{k=1}^{\infty}\binom{n+k-1}{k}\frac{k}{n}\Delta_{1/n}^{k-1}\big(f\circ\varphi^{-1}\big)(0)\left(\frac{x}{1-x}\right)^k \\
&= (1-x)\sum_{k=0}^{\infty}\binom{n+k}{k}\Delta_{1/n}^k\big(f\circ\varphi^{-1}\big)(0)\left(\frac{x}{1-x}\right)^k \\
&\quad + (1-x)\sum_{k=0}^{\infty}\binom{n+k}{k+1}\frac{k+1}{n}\Delta_{1/n}^k\big(f\circ\varphi^{-1}\big)(0)\left(\frac{x}{1-x}\right)^{k+1} \\
&= \sum_{k=0}^{\infty}\binom{n+k}{k}\Delta_{1/n}^k\big(f\circ\varphi^{-1}\big)(0)\left(\frac{x}{1-x}\right)^k.
\end{aligned}$$

Notice that, taking into account (9), the derivatives of Baskakov operators can be written as the Baskakov operator of a related function. That is

$$D^i V_n(g,y) = \frac{1}{(1+y)^i} V_n\left(\prod_{j=1}^{i}(n+ne_1+j-1)\Delta_{1/n}^i g, y\right). \qquad \square \tag{10}$$

Proposition 9. *For $\varphi(x) = x/(1-x)$, $f \in \mathcal{D}[0,1)$ and $H_k := \prod_{j=1}^{k}(n+n\varphi+j)$, $k \in \mathbb{N}$, we have*

$$D_\varphi^k(M_n f)(x) = (1-x)^k M_n\big(H_k\big(\big(\Delta_{1/n}^k\big(f\circ\varphi^{-1}\big)\big)\circ\varphi\big)\big)(x). \tag{11}$$

Proof. We will prove by induction that, for $y \in [0,\infty)$,

$$D^k\big((M_n f)\circ\varphi^{-1}\big)(y) = (1+y)^{-(k+1)} V_n\big[U^{-1}(G_k\circ\varphi)\big](y),$$

where $G_k = (H_k\circ\varphi^{-1})\Delta_{1/n}^k g$ and $g = f\circ\varphi^{-1}$. For $k=1$ we have (see (10) and (8))

$$\begin{aligned}
D\big((M_n f)\circ\varphi^{-1}\big)(y) &= D\left(\frac{1}{1+y}V_n\big(U^{-1}f,y\big)\right) = \frac{1}{(1+y)^2}V_n\big(-U^{-1}f + (n+ne_1)\Delta_{1/n}U^{-1}f, y\big) \\
&= \frac{1}{(1+y)^2}V_n\big[U^{-1}(G_1\circ\varphi)\big](y).
\end{aligned}$$

Assume that (11) holds for k. First, notice that

$$\begin{aligned}
\Delta_{1/n}(G_k)(y) &= \Delta_{1/n}^k g(y+1/n)\prod_{j=1}^{k}(n+ny+j+1) - \Delta_{1/n}^k g(y)\prod_{j=1}^{k}(n+ny+j) \\
&= \Delta_{1/n}^k g(y+1/n)\prod_{j=2}^{k+1}(n+ny+j) - \Delta_{1/n}^k g(y)\prod_{j=1}^{k}(n+ny+j) \\
&= \Delta_{1/n}^{k+1} g(y)\prod_{j=2}^{k+1}(n+ny+j) + k\Delta_{1/n}^k g(y)\prod_{j=2}^{k}(n+ny+j)
\end{aligned}$$

and $(n+n\varphi+1)((\Delta_{1/n}G_k)\circ\varphi) = ((\Delta_{1/n}^{k+1}g)\circ\varphi)H_{k+1} + k((\Delta_{1/n}^{k}g)\circ\varphi)H_k$. Hence

$$\begin{aligned} U^{-1}\big[(n+n\varphi+1)\big((\Delta_{1/n}G_k)\circ\varphi\big)\big] &= (1+e_1)(n+ne_1+1)(\Delta_{1/n}G_k) \\ &= (1+e_1)\big[\big(\big(\Delta_{1/n}^{k+1}g\big)\big(H_{k+1}\circ\varphi^{-1}\big)\big) + k\big(\big(\Delta_{1/n}^{k}g\big)\big(H_k\circ\varphi^{-1}\big)\big)\big] \\ &= (1+e_1)(G_{k+1}+kG_k) \end{aligned}$$

and

$$\begin{aligned} n(1+e_1)\Delta_{1/n}\big(U^{-1}(G_k\circ\varphi)\big) &= U^{-1}\big[(n+n\varphi+1)\big(\Delta_{1/n}(G_k)\circ\varphi\big)\big] + U^{-1}(G_k\circ\varphi) \\ &= (1+e_1)\big(G_{k+1}+(k+1)G_k\big) = U^{-1}\big(\big(G_{k+1}+(k+1)G_k\big)\circ\varphi\big). \end{aligned}$$

Finally, from (8) we obtain

$$\begin{aligned} D^{k+1}(M_n g)(y) &= \frac{V_n(n(1+e_1)\Delta_{1/n}U^{-1}[G_k\circ\varphi], y)}{(1+y)^{k+2}} - \frac{(k+1)V_n(U^{-1}[G_k\circ\varphi)])(y)}{(1+y)^{k+2}} \\ &= -\frac{(k+1)V_n(U^{-1}[G_k\circ\varphi)])(y)}{(1+y)^{k+2}} + \frac{V_n[U^{-1}(G_{k+1}\circ\varphi+(k+1)G_k\circ\varphi), y]}{(1+y)^{k+2}} \\ &= (1+y)^{-(k+2)}V_n\big[U^{-1}(G_{k+1}\circ\varphi), y\big]. \qquad \Box \end{aligned}$$

Now we can conclude our main result.

Theorem 10. *For $\varphi(t) = t/(1-t)$, fix $k \in \mathbb{N}_0 \cup \{\infty\}$ and a function $f \in C^k[0,1) \cap \mathcal{D}[0,1)$.*

(i) *If f is φ-convex of order k, then $M_n f$ is φ-convex of order k.*
(ii) *If $f\circ\varphi^{-1}$ is absolutely monotonic of order k, then $M_n f$ is absolutely monotonic of order k.*
(iii) *If f is completely monotonic of order k, then $M_n f\circ\varphi^{-1}$ is completely monotonic of order k.*

Proof. (i) If f is φ-convex of order k, then for each $x \in [0,1)$ and $h > 0$, $\Delta_h^k(f\circ\varphi^{-1})(\varphi(x)) \geqslant 0$. It follows from (11) that $D^k((M_n f)\circ\varphi^{-1})(\varphi(x)) \geqslant 0$. Hence, $(M_n f)\circ\varphi^{-1}$ is convex of order k. Since $(M_n f)$ is continuous, it follows from Theorem 5 that $M_n f$ is φ-convex of order k. (ii) and (iii) follow immediately from (i) and Theorem 7. $\Box$

Remark. We have obtained a class of functions f such that $M_n f$ is an absolutely monotonic function, for all $n \in \mathbb{N}$. This class is big enough, more precisely, it is a convex cone of functions. It is straightforward to check that each power φ^k of the function $\varphi(t) = t/(1-t)$ belongs to this cone.

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